The dielectric behavior of the living cell suspensions

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Abstract

In the limit of small concentrations and weak applied electric fields, the dielectric permittivity of suspensions of arbitrarily shaped, shelled and charged particles is calculated. It is proved that the dielectric behavior at low frequencies is dominated by the effects of the diffusion of the free charges on the shell surfaces. Our theoretical formula is valid in the low range of frequencies (α dispersion) as well as in the high range of frequencies (β dispersion). Will result that one can measure the membrane electrical potential by a simple investigation of the living cell suspension dielectric properties.

1 Introduction

The development of quantitative methods of characterizing the living matter represents the subject of many researches of our days. The study of the electric (dielectric) properties of biological systems belongs to this kind of programs. A simple measurement of the dielectric permittivity of a living cells suspension provides us with an important set of data which can be used to describe the living matter [4, 17]. There are many factors which may influence the dielectric behavior of the biological materials: structure, molecular dipoles orientation, self interactions, surface conductance, diffusion, membrane transport processes etc. All these factors influence one each other and it is hard (if not impossible) to separate the effect of a single one. However, some effects are dominant in certain ranges of frequencies and certain conditions. For example, in the β-dispersion range (4 MHz-8 MHz), the dielectric behavior of the living cell suspensions is mainly influenced by the dielectric structure. At low frequencies, it is dominated by the diffusion effects of the free charges accumulated around the cell membrane [5]. Also, the orientation of molecular dipoles and membrane transport processes become relevant when strong electric fields are applied. For a survey of the field and an extended bibliography one can consult [9].

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The electric activity, in particular, the existence of the membrane potential is one of the simplest factor which allows us to discern between a living and a nonliving cell. We will show in this paper its influence on the dielectric properties of biological materials. Our model do not include rotational effects or self interactions, so one expects that our results to be valid only for diluted suspensions and weak electric fields. We like to think that our work is complementary to the works done in electrophoresis, where the applied electric fields are very strong and the object of investigations is the induced membrane potential [11]. In this paper, the applied electric fields are weak and the object of investigation is the influence of the membrane potential.

We calculate the dielectric permittivity of suspensions of shelled, arbitrarily shaped particles with a superficial distribution of free charges on the two faces of the shell, in the limit of small concentrations and weak applied electric fields. The method used here is similar with that of [3], the basic tools being the functional calculus and the spectral decomposition of the operators. For the real situation, the distributions of free charges are rather spatial than superficial. For usual values of the (3-D) diffusion constants, they have the support in the immediate vicinity of the membrane (the distributions are practically zero for the distances larger than 10^{-9} m) and this allows us to treat them as superficial distributions. Our numerical application will reveal the dependence of the dielectric permittivity on the membrane electrical potential, mobility of the free distribution of charges and particle geometry. All these factors are equally important in the low range of frequency. It turns out that the consideration of non-spherical geometries (especially non-convex geometries) may play an important role in some situations. For example, if one wants to investigate the living cell cycles by dielectric measurements on synchronized cell suspensions, then definitely, near the subdivision point, the shape effects are considerable [7].

We don't know yet how to relate our results with the electrorotation theory [10] or how to combine them. We consider that is premature to compare our theoretical results with the experimental results obtained by electrorotation, which, at this time, seems to be the only accurate experimental data in the low range of frequencies. This is mainly because of different regimes of the applied electric field and concentrations.

2 The general analysis

From now, the perturbation of the external electric field (which will be called the excitation), due to the presence of a particle, will be called the response of that particle to the excitation. It is known that the Lorentz method of dielectric permittivity calculus works only if the response of the suspension constituents is linear in respect to the excitation. When diffusion effects are considered, this fact is no longer true. This is the reason we start our analysis with the response of a single particle to an external excitation.

Let us consider a particle with two dielectric phases (shelled particle), placed in a spacial homogeneous and temporal oscillating electric field. In addition, we

consider that we have a superficial charge distribution on the shell faces. Laplace equation, satisfied by the electrical potential, has to be completed with the boundary conditions on the separation surfaces of the dielectric phases. These conditions will be deducted from charge conservation equation. If one considers a surface Σ which separates two dielectric mediums, $\mathcal{D}_{\pm}$, with a free charge ρ distributed on it, then the electrical current density is formed by a volume one, given by $\vec{j}^{\pm} = \sigma^{\pm} \cdot \vec{E}$ and a singular density, $\vec{j}_{sg}$, having the support on the interface Σ. Denoting with n the net charge concentration (note that a part of charges are due to different conductivities σ^{+}, σ^{-}), the continuity equation for a domain $\mathcal{D}$, centered on the surface, is

$$-\frac{d}{dt}\int_{\mathcal{D}} n\, dv = \oint_{\partial\mathcal{D}} \vec{j}_{total} d\vec{S} = \oint_{\partial\mathcal{D}} \vec{j}_{vol} d\vec{S} + \oint_{\Gamma} \vec{j}_{sg} d\vec{\Gamma}, \qquad (1)$$

where $\Gamma = \partial\mathcal{D} \cap \Sigma$. Using the Maxwell equation, $n = div\vec{D}$, it follows

$$\oint_{\partial\mathcal{D}} (\sigma \cdot \vec{E} + \frac{\partial}{\partial t}\vec{D}) \cdot d\vec{S} = -\oint_{\Gamma} \vec{j}_{sg} d\vec{\Gamma}. \qquad (2)$$

The singular current is the only cause of the superficial distribution variation

$$-\frac{d}{dt}\int_{\mathcal{D}\cap\Sigma} \rho \cdot dS = \oint_{\Gamma} \vec{j}_{sg} d\vec{\Gamma} \qquad (3)$$

Finally, the passing equation of the electrical field, in the integral form, is

$$\oint_{\partial\mathcal{D}} (\sigma \cdot \vec{E} + \frac{\partial}{\partial t}\vec{D}) \cdot d\vec{S} = \frac{d}{dt}\int_{\mathcal{D}\cap\Sigma} \rho \, dS, \qquad (4)$$

and in the differential form is

$$\vec{n}(\sigma^{+} \cdot \vec{E}^{+} + \frac{\partial}{\partial t}\vec{D}^{+}) - \vec{n}(\sigma^{-} \cdot \vec{E}^{-} + \frac{\partial}{\partial t}\vec{D}^{-}) = \frac{\partial\rho}{\partial t}, \qquad (5)$$

where $\vec{n}$ represents the normal at the interface. For shelled particles, the complete system of equations will be

$$\begin{cases} \Delta\Phi = 0 \; ; \; x \in \Re^{3}\backslash(\Sigma_1 \cup \Sigma_2) \\ \vec{n}(\sigma^{+} \cdot \vec{E}^{+} + \frac{\partial}{\partial t}\vec{D}^{+}) - \vec{n}(\sigma^{-} \cdot \vec{E}^{-} + \frac{\partial}{\partial t}\vec{D}^{-}) = \frac{\partial\rho}{\partial t} \\ x \in \Sigma_1 \cup \Sigma_2 \\ div_{\Sigma}\, \vec{j}_{sg} = -\frac{\partial\rho}{\partial t}; \; x \in \Sigma_1 \cup \Sigma_2 \\ \vec{E} \longrightarrow \vec{E}_0 \cdot \exp(j\omega_0 t) \text{ as } \left|\vec{x}\right| \to \infty \end{cases} \qquad (6)$$

where Σ_1, Σ_2 are the external respective the internal face of the shell and $\vec{E}_0 \exp(j\omega_0 t)$ is the external field.

3

The temporal Fourier decomposition leads to the following form of the equations and boundary conditions:

$$
\begin{cases}
\Delta\Phi = 0 \; ; \; x \in \Re^3 \backslash (\Sigma_1 \cup \Sigma_2) \\
(\sigma^+ + j\omega\varepsilon^+)\overrightarrow{n} \cdot \vec{E}^+ - (\sigma^- + j\omega\varepsilon^-)\overrightarrow{n} \cdot \vec{E}^- = j\omega\rho, \\
x \in \Sigma_1 \cup \Sigma_2 \\
div_\Sigma \overrightarrow{j}_{sg} = -j\omega\rho; \; x \in \Sigma 1 \cup \Sigma_2 \\
\vec{E} \longrightarrow \begin{cases} \vec{E}_0 \; ; \; \omega = \omega_0 \\ 0; \; \omega \neq \omega_0 \end{cases} \quad \text{as} \quad \left|\overrightarrow{x}\right| \to \infty.
\end{cases}
\tag{7}
$$

For $\omega \neq 0$, the equation $div_\Sigma \overrightarrow{j}_{sg} = -j\omega\rho$ imposes

$$
\int_{\Sigma_{1,2}} \rho(\omega)dS = -\frac{1}{j\omega} \int_{\Sigma_{1,2}} div_\Sigma \overrightarrow{j}_{sg}dS = 0.
\tag{8}
$$

This means that for $\omega \neq \omega_0$ and $\omega \neq 0$, we obtain trivial boundary conditions. So the system has a trivial solution. For $\omega = 0$, the above equation cannot be written. It is being replaced with the condition

$$
\int_{\Sigma_{1,2}} \rho(0)dS = \pm Q,
\tag{9}
$$

where $\pm Q$ represent the total free charge distributed on the two interfaces. This leads to nontrivial boundary conditions and consequently to a nontrivial solution. In consequence, the solution of the system has the following general form:

$$
\begin{cases}
\Phi(x,t) = \Phi_0(x) + \Phi(x) \cdot \exp(j\omega_0 t) \\
\rho(x,t) = \rho_0(x) + \rho(x) \cdot \exp(j\omega_0 t)
\end{cases}
\tag{10}
$$

The components Φ_0 and ρ_0 represent the solution of the system for zero external excitation and will represent the equilibrium of the system. The equilibrium will be discussed in the next section. The components Φ and ρ are solutions of the following system:

$$
\begin{cases}
\Delta\Phi = 0 \; ; \; x \in \Re^3 \backslash (\Sigma_1 \cup \Sigma_2) \\
(\sigma^+ + j\omega\varepsilon^+)\dfrac{\partial\Phi^+}{\partial\overrightarrow{n}} - (\sigma^- + j\omega\varepsilon^-)\dfrac{\partial\Phi^-}{\partial\overrightarrow{n}} = j\omega\rho; \; x \in \Sigma_1 \cup \Sigma_2 \\
div_\Sigma \overrightarrow{j}_{sg} = -j\omega\rho; \; x \in \Sigma_1 \cup \Sigma_2 \\
\vec{E} \longrightarrow \vec{E}_0 \text{ as } \left|\overrightarrow{x}\right| \to \infty.
\end{cases}
\tag{11}
$$

where the pulsation of the excitation was re-denoted with ω. Introducing the expression of the singular current

$$
\overrightarrow{j}_{sg} = \overrightarrow{j}_{conduction} + \overrightarrow{j}_{diffusion} = -\gamma_i \cdot \vec{\nabla}_{\Sigma_i}\Phi - D_i\vec{\nabla}_{\Sigma_i}\rho_i, \; i = 1,2,
\tag{12}
$$

4

and working with the complex electric permittivity $\varepsilon^* = \varepsilon + \sigma/(j\omega)$, we can write

$$
\begin{cases}
\Delta\Phi = 0 \ ; \ x \in \Re^3 \backslash (\Sigma_1 \cup \Sigma_2) \\
\varepsilon_{i-1}^* \dfrac{\partial \Phi^+}{\partial \overrightarrow{n}} - \varepsilon_i^* \dfrac{\partial \Phi^-}{\partial \overrightarrow{n}} = \rho_i; \ x \in \Sigma_i \\
div_{\Sigma_i}[-\gamma_i \cdot \vec{\nabla}_{\Sigma_i}\Phi - D_i\vec{\nabla}_{\Sigma_i}\rho_i] = -j\omega\rho_i; \ x \in \Sigma_i \\
\overrightarrow{E} \longrightarrow \overrightarrow{E}_0 \ \text{as} \ \left|\overrightarrow{x}\right| \to \infty.
\end{cases}
\tag{13}
$$

where the index Σ means that the operators are calculated on the surface. The system is not linear, because the conductivities of the superficial charges, γ_i, depend on ρ. We will consider in the following that we are in the limit of weak external electric fields so we can consider that the conductivities of the free charges are given by the equilibrium configuration. Thus the system becomes linear. Now, the total electric field will be the sum of the electrical fields of the equilibrium configuration and of the perturbation, $\overrightarrow{E}_{total} = \overrightarrow{E}_e + \overrightarrow{E}$. In the weak fields approximation, the average on the different orientations of the particle leads to the following value of the electric field inside of the particle:

$$
\left\langle \overrightarrow{E} \right\rangle_{orientations} = \overrightarrow{E}_0 \cdot \frac{1}{4\pi} \int_{\Omega_{\overrightarrow{N}}} \overrightarrow{E}_{\overrightarrow{N}} \cdot \overrightarrow{N} \ d\Omega_{\overrightarrow{N}},
\tag{14}
$$

where $\overrightarrow{E}_{\overrightarrow{N}}$ is the electric field (only the perturbation part) inside of the particle, when the external electric field, $\overrightarrow{N}$, is of norm one. Further, this average can be calculate with the formula

$$
\left\langle \overrightarrow{E} \right\rangle_{orientations} = \frac{1}{3}\left(\sum_i \overrightarrow{N}_i \cdot \overrightarrow{E}_{\overrightarrow{N}_i}\right)\overrightarrow{E}_0,
\tag{15}
$$

where i designates three orthogonal directions. The average electric field is proportional with the excitation. This allows us to use the Lorenz formula

$$
\varepsilon_{sus} = \varepsilon_o\left(1 + \frac{p\alpha}{1 - \frac{p\alpha}{3}}\right)
\tag{16}
$$

for the suspension dielectric permittivity calculus. Here p is the volume concentration of the suspension and α is the polarization of the particles

$$
\alpha = \frac{1}{VE_0}\int_V dv \cdot \frac{\varepsilon - \varepsilon_o}{\varepsilon_o}\left\langle \overrightarrow{E} \right\rangle_{orientations}
\tag{17}
$$

where V is the particle volume, ε_o is the dielectric permittivity of the exterior medium and ε is the dielectric permittivity of the particle.

3   The equilibrium

In the following the primitivities and the conductivities will be denoted by: ε_0, σ_0 for the exterior, ε_1, σ_1 for the membrane, ε_2, σ_2 for the interior. The system which establish the equilibrium configuration is obtained from the general system by cancelling the temporal derivatives:

$$\begin{cases} \Delta\Phi_e = 0 \; ; \; x \in \Re^3 \backslash (\Sigma_1 \cup \Sigma_2) \\ \sigma_{i-1} \dfrac{\partial\Phi_e^+}{\partial\vec{n}} \mid_{\Sigma_i} = \sigma_i \dfrac{\partial\Phi_e^-}{\partial\vec{n}} \mid_{\Sigma_i} \; ; \; i = 1, 2 \\ div_{\Sigma_i} \vec{j}_{sg} = 0 \\ \int_{\Sigma_{1,2}} \rho_0 dS = \pm Q. \end{cases} \tag{18}$$

If we consider the conductivity of the membrane, σ_1, equal with zero, we obtain trivial Neumann condition for the electric potential, inside and outside of the particle. This imposes constant electrical potential in the two regions. The distribution of charges on the two interfaces, $\tau_{1,2}$, can be calculated by applying the Gauss law:

$$\tau_1 = \varepsilon_1 \frac{\partial\Phi^-}{\partial\vec{n}} \mid_{\Sigma_1}; \;\; \tau_2 = \varepsilon_1 \frac{\partial\Phi^+}{\partial\vec{n}} \mid_{\Sigma_2} \tag{19}$$

where τ includes both, the free charge ρ and the charges which are accumulated on the interfaces due to the conduction currents. The equations for free charges are

$$div[\gamma_i \cdot \vec{E}_t - D\nabla\rho_{0_i}] = 0 \Longrightarrow D\Delta\rho_{0_i} = 0 \Longleftrightarrow \rho_{0_i} = const., \tag{20}$$

where $\vec{E}_t$ is the tangent electric field to $\Sigma_{1,2}$. The link between ρ_0 and Q results by the reason that the inside region is not an electric charges reservoir, which means that the total charge on the interfaces is given only by the free charges. Thus

$$\pm Q = \int_{\Sigma_{1,2}} \rho_{0_{1,2}} dS = \rho_{0_{1,2}} \cdot S_{1,2} \; ; \; Q = C \cdot \Delta V_0 \tag{21}$$

where ΔV_0 is the electric potential difference between the two surfaces (membrane potential) and C is the capacity of a condenser having the geometry of the shell and dielectric permittivity equal with ε_1. The values of the ρ_{0_i} fix the electric conductivity on the two interfaces

$$\gamma_{1,2} = \rho_{0_{1,2}} \cdot u_{1,2} = \pm u_{1,2} \frac{C}{S_{1,2}} \cdot \Delta V_0, \tag{22}$$

where u_i, $i = 1$, 2 are the charge mobilities on the two interfaces.

It was seen that, in the calculus of the dielectric permittivity, only the perturbation part of the electric field is important. So the only influence of the membrane electric potential on the dispersions curves of the dielectric permittivity comes from the above formula.

4 The effective calculus of the polarization

The charges on the second interface lie behind the shell, so the external electric field will have a smaller influence on them than on the charges of the first interface. In the same time, it is an experimental fact that the mobility of the inside charges is much smaller than that of external charges. Both reasons justify the idea of considering the free charges of the inside interface to be fixed (i.e. zero mobility). With this simplification, the value of ρ_2 is fixed at the equilibrium value. Now we have inhomogeneous particles with free electric charges only on their surface.

4.1 The equivalence between an inhomogeneous particle and a homogeneous one

Let us consider first the situation of a homogeneous particle, with dielectric permittivity ε and conductivity σ, placed in the electric field $\vec{E}_0 e^{j\omega t}$. If we use the expression of the simple layer for the electric potential

$$\Phi(\vec{x}) = -\vec{x} \cdot \vec{E}_0 + \frac{1}{4\pi} \cdot \int_{y \in \Sigma} \frac{\mu(\vec{y})}{\left|\vec{x} - \vec{y}\right|} dS_y, \tag{23}$$

the passing equation for the electric field through the surface of the particle, Σ, becomes the following integral equation for the charge distribution μ:

$$\frac{1}{2\lambda}\mu(\vec{x}) - \hat{E}[\mu](\vec{x}) = \vec{n} \cdot \vec{E}_0, \tag{24}$$

where $\lambda = \dfrac{\varepsilon^* - \varepsilon_o^*}{\varepsilon^* + \varepsilon_o^*}$ and $\hat{E}$ is the operator

$$\hat{E}\left[\mu\right]\left(\vec{x}\right) = \frac{1}{4\pi} \cdot \int_{y \in \Sigma} \frac{(\vec{x} - \vec{y}) \cdot \vec{n}_x}{\left|\vec{x} - \vec{y}\right|^3}\mu(\vec{y})dS_y. \tag{25}$$

Using the spectral decomposition of the $\hat{E}$ operator, the solution of this equation is

$$\mu = \sum_n \frac{\lambda}{\frac{1}{2} - \lambda \cdot \chi_n} \cdot \hat{P}_n[\vec{n} \cdot \vec{E}_0], \tag{26}$$

where $\hat{P}_n$ is the spectral projector corresponding to the eigenvalue χ_n. The $\hat{E}$ operator is not a symmetric one, so some precautions are needed. About this operator and the validity of the above decomposition, one can consult [15].

Let us consider now the shelled particle placed in the same electric field. We consider here that the external surface of the shell is obtained by expanding the internal surfaces by a factor $\delta > 1$. The equivalence problem between this inhomogeneous dielectric particle and a homogeneous one can be formulated in

the following terms: there exists a distribution of charges only on the external surface of the shell, μ_e, which provides the true electric field outside of particle. In the dipole approximation, this distribution is given by the following expression:

$$\mu_e = \sum_n \frac{\lambda_n}{\frac{1}{2} - \lambda_n \cdot \chi_n} \cdot \hat{P}_n[\vec{n} \cdot \vec{E}_0]. \tag{27}$$

where λ_n are given by:

$$\lambda_n = \frac{\varepsilon_n^* - \varepsilon_o^*}{\varepsilon_n^* + \varepsilon_o^*}; \ \varepsilon_n^* = \varepsilon_1^* \frac{\chi_n + \frac{1}{2} + \delta^3(\frac{1}{2\lambda_2} - \chi_n)}{\chi_n - \frac{1}{2} + \delta^3(\frac{1}{2\lambda_2} - \chi_n)}; \ \lambda_2 = \frac{\varepsilon_2^* - \varepsilon_1^*}{\varepsilon_2^* + \varepsilon_1^*}. \tag{28}$$

Comparing with the formula for the homogeneous particle, we conclude that the shelled particle is equivalent to a homogeneous particle which has the same geometry and volume. The equivalation relation is given by

$$\varepsilon^* \to \{\varepsilon_n^*\}_n, \text{ or } \varepsilon^* = \sum_n \varepsilon^* \hat{P}_n \to \sum_n \varepsilon_n^* \hat{P}_n. \tag{29}$$

In the particular case of the sphere, the above formula is identical to the existent formula [12]. We will give in the following a sketch of the proof. For the shelled particle, the single layer expression of the potential is

$$\Phi(x) = -\vec{x}E_0 +$$

$$\frac{1}{4\pi} \int_{y \in \Sigma_1} \frac{\mu_1(\vec{y})}{\left|\vec{x} - \vec{y}\right|} dS_y + \frac{1}{4\pi} \int_{y \in \Sigma_2} \frac{\mu_2(\vec{y})}{\left|\vec{x} - \vec{y}\right|} dS_y. \tag{30}$$

The electric potential created by μ_e will be denoted with Φ_e. The potential Φ_e is equal outside of particle with the exact potential, Φ, only if the two potentials fulfill the same Neuman boundary conditions: $\dfrac{\partial \Phi_e^+}{\partial \vec{n}} = \dfrac{\partial \Phi^+}{\partial \vec{n}}$. Using the single layer expression, the above condition can be written as

$$\frac{1}{2}\mu_e - \hat{E}[\mu_e] = \frac{1}{2}\mu_1 - \hat{E}_{11}[\mu_1] - \hat{E}_{12}[\mu_2], \tag{31}$$

where

$$\hat{E}_{ij}[\mu_j] = \frac{1}{4\pi} \cdot \int_{y \in \Sigma_j} \frac{(\vec{x} - \vec{y}) \cdot \vec{n}_x}{\left|\vec{x} - \vec{y}\right|^3} \cdot \mu_j(\vec{y}) dS_y; \ x \in \Sigma_i. \tag{32}$$

The passing equations of the exact electric field, Φ, through the shell surfaces take the form

$$\begin{cases} \frac{1}{2\lambda_1}\mu_1 - \hat{E}_{11}[\mu_1] - \hat{E}_{12}[\mu_2] = \vec{n}\vec{E}_0 \\ \frac{1}{2\lambda_2}\mu_2 - \hat{E}_{21}[\mu_1] - \hat{E}_{22}[\mu_2] = \vec{n}\vec{E}_0 \end{cases} \tag{33}$$

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where $\lambda_i = \dfrac{\varepsilon_i^* - \varepsilon_{i-1}^*}{\varepsilon_i^* + \varepsilon_{i-1}^*}$. Here comes the dipole approximation:

$$\begin{cases} \hat{E}_{12}\left[\mu_2\right] \approx \frac{1}{\delta^3}\hat{E}_{22}\left[\mu_2\right] \\ \hat{E}_{21}\left[\mu_1\right] \approx \hat{E}_{11}\left[\mu_1\right] \end{cases} \tag{34}$$

which says that the normal component of the electric field created by the charge distribution μ_2 on the surfaces Σ_1 has the same angular dependence as on the surfaces Σ_2 but the strength of the field is diminished by a factor δ^{-3}, and that the normal component of the electric field created by the distribution μ_1 on the surface Σ_2 is equal with the normal component on the surface Σ_1. With the observation that the operator $\hat{E}$ is scale invariant, the above equations become

$$\begin{cases} \frac{1}{2\lambda_1}\mu_1 - \hat{E}\left[\mu_1\right] - \frac{1}{\delta^3}\hat{E}\left[\mu_2\right] = \vec{n}\vec{E}_0 \\ \frac{1}{2\lambda_2}\mu_2 - \hat{E}\left[\mu_1\right] - \hat{E}\left[\mu_2\right] = \vec{n}\vec{E}_0 \end{cases} \tag{35}$$

Using again the spectral decomposition of the operator $\hat{E}$ and the link (31) between μ_e and μ_1, μ_2, the desired form of the distribution μ_e follows immediately. One can see that the method can be extended to the multi-shelled particles. In this way, the results of [1] can be extended to arbitrary geometries.

Now, we want to consider the situation when the free distribution of charges ρ is present. For a homogeneous dielectric particle with dielectric permittivity ε and conductivity σ which has a free charge on its surface, the equations are

$$\begin{cases} \varepsilon_o^*\left(\vec{n}\cdot\vec{E}_0 + \frac{1}{2}\mu + E[\mu]\right) - \varepsilon^*\left(\vec{n}\cdot\vec{E}_0 - \frac{1}{2}\mu + E[\mu]\right) = \rho \\ \left[\Delta_\Sigma + \frac{j\omega}{D}\right]\rho = -\frac{\gamma}{D}\Delta_\Sigma\Phi \end{cases} \tag{36}$$

where Δ_Σ is the Laplace operator on the particle surface. Here we have continued the idea of the second section by introducing the simple layer expression of the electric potential. We can pass to the situation of a shelled particle by using the equivalence relation $\varepsilon^* \rightarrow \sum_n \varepsilon_n^* \hat{P}_n$. (Here we use our assumption that the distribution of the free charges of the internal shell surface can be considered immobile). The equations become

$$\begin{cases} \varepsilon_o^*\left(\vec{n}\cdot\vec{E}_0 + \frac{1}{2}\mu + E[\mu]\right) \\ -\left(\sum_n \varepsilon_n^*\cdot\hat{P}_n\right)\left(\vec{n}\cdot\vec{E}_0 - \frac{1}{2}\mu + E[\mu]\right) = \rho \\ \left(\Delta_{\Sigma_1} + \frac{j\omega}{D}\right)\rho = -\frac{\gamma}{D}\Delta_{\Sigma_1}\Phi \end{cases} \tag{37}$$

Here, ρ denotes the free distribution of charges of the external shell surface. One observes that the single layer distribution can be decomposed into $\mu = \mu_\beta + \mu_\alpha$, where μ_β solves the equations for $\rho = 0$. It turns out that μ_β is practically constant at low frequencies and that μ_α is practically zero at high frequencies. In consequence, the μ_β distribution is responsible for the behavior at high frequencies (the range of the beta dispersion) and the addition of μ_α

will give us the behavior at low frequencies (the range of alpha dispersion). The equations for μ_α and ρ will be:

$$\begin{cases} \sum_n \left(\frac{1}{2}(\varepsilon_n^* + \varepsilon_o^*) - \chi_n(\varepsilon_n^* - \varepsilon_o^*)\right) \hat{P}_n \, \mu_\alpha = \rho \\ \left(\Delta + \frac{j\omega}{D}\right) \rho = -\frac{\gamma}{D}\Delta\Phi_\alpha - \frac{\gamma}{D}\Delta\Phi_\beta \end{cases} \tag{38}$$

where Φ_β (Φ_α) represents the electric potential generate by the μ_β (μ_α) distribution. The spectral decomposition of the Laplace operator leads to the following equation for μ_α:

$$\sum_n \left(\frac{1}{2}(\varepsilon_n^* + \varepsilon_o^*) - \chi_n(\varepsilon_n^* - \varepsilon_o^*)\right) \hat{P}_n \, \mu_\alpha$$
$$+\frac{\gamma}{D}\sum_i \frac{\xi_i}{\xi_i + j\omega}\hat{G}_i\tilde{\Phi} = -\frac{\gamma}{D}\sum_i \frac{\xi_i}{\xi_i + j\omega}\hat{G}_i\Phi, \tag{39}$$

where $\hat{G}_i$ is the spectral projector of the Laplace operator corresponding to the eigenvalue ξ_i. Denoting with $\hat{K}$ the operator

$$(\hat{K}\mu)(\vec{x}) = \frac{1}{4\pi} \int\limits_{y\in\Sigma} \frac{\mu(\vec{y})}{|\vec{x} - \vec{y}|} dS_y, \tag{40}$$

the equation takes the form:

$$\begin{cases} \frac{D}{\gamma}\sum_n \left(\frac{1}{2}(\varepsilon_n^* + \varepsilon_o^*) - \chi_n(\varepsilon_n^* - \varepsilon_o^*)\right) \hat{P}_n \\ +\sum_i \frac{\xi_i}{\xi_i + j\omega}\hat{G}_i \circ \hat{K} \end{cases} \mu_\alpha = -\sum_i \frac{\xi_i}{\xi_i + j\omega}\hat{G}_i \circ \left(-\vec{x}\cdot\vec{N} + \hat{K}\mu_\beta\right). \tag{41}$$

We will discuss later our method of solving this equation.

4.2 The formula for polarization

We can use the idea of equivalence to find a compact formula for polarization. For this, we have to transform the volume integral which appears in the definition of α,

$$\alpha = \frac{1}{V} \int d\Omega_{\vec{N}}\int\limits_V dv \cdot \frac{\varepsilon - \varepsilon_o}{\varepsilon_o}\vec{N}\cdot\vec{E}, \tag{42}$$

into an integral on the external side of the particle surface where we can use the equivalence. In the above formula, ε depends on the point, and $\vec{E}$ is the exact field inside of the particle, prodeced by the excitation $\vec{N}$, of norm one. We will calculate the term as follow:

$$\int\limits_V dv \cdot \frac{\varepsilon - \varepsilon_o}{\varepsilon_o}\vec{N}\cdot\vec{E} = \int\limits_V dv \left\{\frac{1}{\varepsilon_o}grad\left(\vec{x}\cdot\vec{N}\right)\cdot\vec{D} + div\left(\vec{N}\Phi\right)\right\}$$
$$= \frac{1}{\varepsilon_o}\int\limits_V dv\, div\left((\vec{x}\cdot\vec{N})\cdot\vec{D}\right) + \int\limits_V dv\, div\left(\vec{N}\Phi\right) \tag{43}$$
$$= \frac{1}{\varepsilon_o}\int\limits_{\Sigma^-} (\vec{x}\cdot\vec{N})\vec{D}\cdot d\vec{S} + \int\limits_{\Sigma^-} \Phi\vec{N}\cdot d\vec{S}$$

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where, if it is used the passing relations for $\vec{D}$, $D_n^+ - D_n^- = \rho$ and the continuity of the potential, the integrals can be processed on the external side of the surface.

$$
\begin{aligned}
&\int_V dv \cdot \frac{\varepsilon - \varepsilon_o}{\varepsilon_o} \vec{N} \cdot \vec{E} = \frac{1}{\varepsilon_o} \int_{\Sigma+} (\vec{x} \cdot \vec{N})(-\rho + D_n^+) \cdot dS \\
&+ \int_{\Sigma+} \Phi \vec{N} \cdot d\vec{S} = \frac{1}{\varepsilon_o} \int_{\Sigma+} (\vec{x} \cdot \vec{N})(-\rho + \varepsilon_o E_n^+) \cdot dS + \int_{\Sigma+} \Phi \vec{N} \cdot d\vec{S}.
\end{aligned}
\tag{44}
$$

Outside of particle, the electric field is equal to that of the equivalent homogeneous particle. The normal component of the electric field can be expressed with the help of the $\hat{E}$ operator:

$$
E_n^+ = -\frac{\partial \Phi^+}{\partial \vec{n}} = \vec{n} \cdot \vec{N} + \frac{1}{2}\mu + \hat{E}\mu,
\tag{45}
$$

where μ is the equivalating charge distribution (solution of (37)), resulting:

$$
\begin{aligned}
&\int_V dv \cdot \frac{\varepsilon - \varepsilon_o}{\varepsilon_o} \vec{N} \cdot \vec{E} = \frac{1}{\varepsilon_o} \int_{\Sigma+} (\vec{x} \cdot \vec{N}) \left(-\rho + \varepsilon_o \left(\vec{n} \cdot \vec{N} + \frac{1}{2}\mu + \hat{E}\mu \right) \right) dS \\
&- \int_V \vec{N} \cdot \vec{E}_e\, dv = \frac{1}{\varepsilon_o} \int_{\Sigma-} (\vec{x} \cdot \vec{N}) \left(\sum_n \varepsilon_n^* \cdot \hat{P}_n \right) \left(\vec{n} \cdot \vec{N} - (\tfrac{1}{2} - \chi_n)\mu \right) dS \\
&- \int_{\Sigma-} (\vec{x} \cdot \vec{N}) \vec{E}_e d\vec{S}
\end{aligned}
\tag{46}
$$

Having that $\vec{n}\vec{E}_e^- = \frac{\partial \Phi_e}{\partial \vec{n}} = \vec{n} \cdot \vec{N} - \frac{1}{2}\mu + \hat{E}[\mu]$, we can continue

$$
\begin{aligned}
&\int_V dv \cdot \frac{\varepsilon - \varepsilon_o}{\varepsilon_o} \vec{N} \cdot \vec{E} \\
&= \int_{\Sigma} (\vec{x} \cdot \vec{N}) \left(\sum_n \frac{(\varepsilon_n^* - \varepsilon_o)}{\varepsilon_o} \cdot \hat{P}_n \right) \left(\vec{n} \cdot \vec{N} - (\tfrac{1}{2} - \chi_n)\mu \right) dS \\
&= \sum_n \frac{(\varepsilon_n^* - \varepsilon_o)}{\varepsilon_o} \cdot \int_{\Sigma} (\vec{x} \cdot \vec{N}) \hat{P}_n \left(\vec{n} \cdot \vec{N} - (\tfrac{1}{2} - \chi_n)\mu \right) dS.
\end{aligned}
\tag{47}
$$

If one denotes the scalar product on the $L^2(\Sigma)$ functions space by $\langle\,,\,\rangle$,

$$
\langle \varphi, \phi \rangle = \int_{\Sigma} \bar{\varphi}(\vec{x}) \phi(\vec{x})\, dS_x,\ \forall\ \varphi,\ \phi \in L^2(\Sigma),
\tag{48}
$$

we obtain

$$
\begin{aligned}
&\int_V dv \cdot \frac{\varepsilon - \varepsilon_o}{\varepsilon_o} \vec{N} \cdot \vec{E} = \sum_n \frac{(\varepsilon_n^* - \varepsilon_o^*)}{\varepsilon_o^*} \times \\
&\left\{ \left\langle \vec{x} \cdot \vec{N} \mid \hat{P}_n \mid \vec{n} \cdot \vec{N} \right\rangle - (\tfrac{1}{2} - \chi_n) \left\langle \vec{x} \cdot \vec{N} \mid \hat{P}_n \mid \mu \right\rangle \right\}
\end{aligned}
\tag{49}
$$

Having in view the discussion about the average on the orientations, the angular integral can be eliminated and the final formula for polarization becomes

$$\alpha = \frac{1}{3V}\sum_{n,i}\frac{(\varepsilon_n^* - \varepsilon_o^*)}{\varepsilon_o^*}\left\{\left\langle\vec{x}\cdot\vec{N_i}\mid\hat{P}_n\mid\vec{n}\cdot\vec{N_i}\right\rangle \right.$$
$$\left. -(\tfrac{1}{2}-\chi_n)\left\langle\vec{x}\cdot\vec{N_i}\mid\hat{P}_n\mid\mu^i\right\rangle\right\},$$

(50)

where $i = 1, 2, 3$ denotes three orthogonal directions and μ^i denotes the total distribution of the single layer corresponding at the excitation $\vec{N_i}$. If the free charge is not present, the above formula becomes

$$\alpha = \frac{1}{3V}\sum_{n,i}\frac{(\varepsilon_n^* - \varepsilon_o^*)}{\varepsilon_o^*}\left\{\left\langle\vec{x}\cdot\vec{N_i}\mid\hat{P}_n\mid\vec{n}\cdot\vec{N_i}\right\rangle\right.$$
$$\left. -(\tfrac{1}{2}-\chi_n)\left\langle\vec{x}\cdot\vec{N_i}\mid\frac{\lambda_n}{\frac{1}{2}-\chi_n\lambda_n}\hat{P}_n\mid\vec{n}\cdot\vec{N_i}\right\rangle\right\}$$
$$= \frac{1}{3V}\sum_{n,i}\frac{\lambda_n}{\frac{1}{2}-\chi_n\lambda_n}\left\langle\vec{x}\cdot\vec{N_i}\mid\hat{P}_n\mid\vec{n}\cdot\vec{N_i}\right\rangle.$$

(51)

This formula is exact in the high frequencies range and it describes correctly the β dispersion region. The exact formula of α must be completed with the term which comes from μ_α

$$\alpha = \frac{1}{3V}\sum_{n,i}\frac{\lambda_n}{\frac{1}{2}-\chi_n\lambda_n}\left\langle\vec{x}\cdot\vec{N_i}\mid\hat{P}_n\mid\vec{n}\cdot\vec{N_i}\right\rangle$$
$$-\frac{1}{3V}\sum_{n,i}\frac{(\frac{1}{2}-\chi_n)(\varepsilon_n^* - \varepsilon_o^*)}{\varepsilon_o^*}\left\langle\vec{x}\cdot\vec{N_i}\mid\hat{P}_n\mid\mu_\alpha^i\right\rangle$$

(52)

As we already discussed, the last term is practically zero at high frequencies and becomes dominant at low frequencies.

5 Numerical results

First, we will present our method of solving the complicated equation (41) of μ_α. Here, the principal problem is the calculation of the spectral projectors of $\hat{E}$ and Δ_Σ operators. Let us suppose that our surface is given in the spherical coordinates by $r = r(\theta, \varphi)$. To solve the problem for the $\hat{E}$ operator, we will choose the following orthonormal basis in $L^2(\Sigma)$:

$$\{\mathcal{Y}_{lm}\}_{\substack{m\in\mathbf{Z}\\l\geq|m|}}; \mathcal{Y}_{lm}(\theta, \varphi) = \frac{Y_{lm}(\theta, \varphi)}{\sqrt{\sigma(\theta, \varphi)}},$$

(53)

where $\{Y_{lm}\}_{l,m}$ represents the spherical harmonics and $dS = \sigma(\theta, \varphi)\sin(\theta)\,d\theta d\varphi$. The first step is the calculation of the matrix elements of the operator $\hat{E}$. Then, taking acceptable truncated matrices, we will calculate the approximative spectrum. The approximative projectors will be given by $\hat{P}_n = P_n\left(\hat{E}\right)$, where P_n

are the polynomials which satisfy $P_n (\chi_m) = \delta_{nm}$ for each eigenvalue χ_m. For the operator Δ_Σ, we have followed the way presented in [14].

We have chosen the surfaces of figure 1 for our numerical application. As we already emphasized, γ is proportional with the electrical potential of the membrane and the mobility of the free charges. Because the diffusion constant is also proportional with the superficial charge mobility, the quantity γ/D is proportional only to the membrane electrical potential. We will choose different values for the diffusion coefficient, D, and for the ratio γ/D, which will give us the dependance of the dispersion curves on the mobility of free charges and membrane electrical potential. Because the α effect on the conductivity is much smaller than on the dielectric constant, we are enforced to consider two different values of the parameters: one for which one can see in detail the behavior of dielectric constant and one for which one can see in detail the behavior of conductivity. The results are presented in figures 2-13. The following parameters were kept constant in our numerical analysis to the following values (the units are in S.I): $\varepsilon_0 = 78\varepsilon_{vac}$, $\varepsilon_1 = 10\varepsilon_{vac}$, $\varepsilon_2 = 50\varepsilon_{vac}$, $\sigma_0 = 0.2$, $\sigma_1 = 0$, $\sigma_2 = 0.3$, $p = 0.06$, $\delta = 1.004$.

6   Conclusions

We want to point out first the relation of our results with the existent results. In the β-dispersion range of frequencies, the importance of $\hat{E}$ operator, in this type of analysis, was pointed out in [18]. Basically, this method reduces the original equations of the electric field to surface equations. Mathematically, this means that it transforms a problem involving unbounded operators (Laplace operator corresponding to certain boundary conditions) to a problem involving compact operators (the operator $\hat{E}$). Here and in [6] it is given a semi-empirical justification of the formulas (28) but, unfortunately, the formula of polarization is not exact. Our paper states in very precise terms what equivalation means, proves rigorously the formulas (28) and that of polarization. Another result directly related with our approach would be [2], where the method of equivalation is discussed. Note that this result (and the method itself) depends on very particular surfaces considered there. In the α-dispersion range of frequencies, there are some results for the case of spherical geometry [5] (and what follows after) but, as is stated in the beginnig of this paper, the results are valid only in a quasi-statical regime. One can see that, in the limit of zero diffusion constants, its passing equations does not reduce to the usual equation, but rather to two separate equations: one for conduction current and one for displacement current. In fact, this was our reason for we have treated very carefully the introductory sections.

In conclusion, this approach gives a unitary treatment of α and β-dispersion in the general context of arbitrary geometries. In the β-dispersion range of frequency, our paper corrects and puts some existent results in rigorous settings. In the α range, it proposes a completely new approach which allows us to avoid problems like that mentioned before. Also, it proves the general formula of po-

larization which includes the effects of the free charges. Our model reproduces, qualitatively, the dielectric behavior of living cell suspensions in both α and β frequencies ranges and is flexible enough to reproduce any given curve which has this shape. The numerical application shows a very strong dependency of the dielectric dispersion curves on the membrane electrical potential. This fact anticipates the possibility of the experimental measurement of the membrane electrical potential by a simple measurement of the dielectric permittivity. Besides this important cell parameter, one can measure the mobility of the free charges and, as in [13], the volume concentration of the suspension. We emphasize that this method is a non-destructive and can be a very fast one (a discussion about this can be found in [8]).

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Figure captions

Fig. 1 The surfaces chosen for our numerical application:
$r(\theta) = (1 + a\cos\theta)/(1 + a)$, where $a = 1, 2, 3$.

Fig. 2 The dependance of dielectric permittivity on membrane potential. The numerical values are $a = 1$, $D = 10^{-8}$, $\gamma/D = 1$ (plus), $\gamma/D = 3$ (circle) and $\gamma/D = 5$ (star).

Fig. 3 The dependance of conductivity on membrane potential. The numerical values are $a = 1$, $D = 10^{-8}$, $\gamma/D = 1$ (plus), $\gamma/D = 3$ (circle) and $\gamma/D = 5$ (star).

Fig. 4 The dependance of dielectric permittivity on membrane potential. The numerical values are $a = 1$, $D = 10^{-8}$, $\gamma/D = 0.1$ (plus), $\gamma/D = 0.5$ (circle) and $\gamma/D = 1$ (star).

Fig. 5 The dependance of conductivity on membrane potential. The numerical values are $a = 1$, $D = 10^{-8}$, $\gamma/D = 0.1$ (plus), $\gamma/D = 0.5$ (circle) and $\gamma/D = 1$ (star).

Fig. 6 The dependance of dielectric permittivity on mobility. The numerical values are $a = 1$, $\gamma/D = 5$, $D = 10^{-7}$ (plus), $D = 10^{-8}$ (circle) and $D = 10^{-9}$ (star).

Fig. 7 The dependance of conductivity on mobility. The numerical values are $a = 1$, $\gamma/D = 5$, $D = 10^{-7}$ (plus), $D = 10^{-8}$ (circle) and $D = 10^{-9}$ (star).

Fig. 8 The dependance of dielectric permittivity on mobility. The numerical values are $a = 1$, $\gamma/D = 0.5$, $D = 10^{-7}$ (plus), $D = 10^{-8}$ (circle) and $D = 10^{-9}$ (star).

Fig. 9 The dependance of conductivity on mobility. The numerical values are $a = 1$, $\gamma/D = 0.5$, $D = 10^{-7}$ (plus), $D = 10^{-8}$ (circle) and $D = 10^{-9}$ (star).

Fig. 10 The dependance of dielectric permittivity on geometry. The numerical values are $\gamma/D = 0.5$, $D = 10^{-8}$, $a = 1$ (plus), $a = 2$ (circle) and $a = 3$ (star).

Fig. 11 The dependance of conductivity on geometry. The numerical values are $\gamma/D = 0.5$, $D = 10^{-8}$, $a = 1$ (plus), $a = 2$ (circle) and $a = 3$ (star).

Fig. 12 The dependance of dielectric permittivity on geometry. The numerical values are $\gamma/D = 5$, $D = 10^{-8}$, $a = 1$ (plus), $a = 2$ (circle) and $a = 3$ (star).

Fig. 13 The dependance of conductivity on geometry. The numerical values are $\gamma/D = 5$, $D = 10^{-8}$, $a = 1$ (plus), $a = 2$ (circle) and $a = 3$ (star).

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